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State Transitions as Morphisms for Complete Lattices†

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We enlarge the hom-sets of categories of complete lattices by introducing 'state transitions' as generalized morphisms. The obtained category is then compared with a functorial quantaloidal enrichment and a contextual quantaloidal enrichment that uses a specific concretization in the category of sets and partially defined maps ($\mathcal{P}$*arset*).

1. INTRODUCTION

In this paper we present a construction that abstracts the concept of 'state transition' as introduced in Amira *et al.* (1998) and Coecke and Stubbe (1999a, b), making it applicable to any subcategory $\mathcal{A}$ of $\mathcal{TCL}at$. We compare this construction, the result of which is a quantaloid that we call $\mathfrak{Q}^{st}\mathcal{A}$, with two other quantaloidal extensions that arise naturally when considering the action of the power-functor on $\mathcal{A}$. In fact, one of these natural extentions is functorial, we denote it by $\mathfrak{Q}^{-}\mathcal{A}$, and the other, called $\mathfrak{Q}^{+}\mathcal{A}$, is contextual in the sense that its construction relies entirely on the fact that $\mathcal{A}$ is a $\mathcal{P}$*arset*-concrete category. The main result of this paper is then that in all nontrivial cases $\mathfrak{Q}^{st}\mathcal{A}$ lies strictly between $\mathfrak{Q}^{-}\mathcal{A}$ and $\mathfrak{Q}^{+}\mathcal{A}$.

Applying the construction $\mathfrak{Q}^{st}$ to the category $\mathcal{P}$*rop*, which was introduced in Moore (1995, 1999), reveals that the latter has to be 'enriched' in order to constitute an appropriate mathematical object for defining 'state transitions'. However, it must be noted that the physical inspiration for our

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categories is essentially different from Moore's: $\mathcal{P}rop$ is constructed to express the equivalence of a property lattice and a state space description for a physical system as a categorical equivalence of $\mathcal{P}rop$ and $\mathcal{S}tate$, inspired by a similar situation for categories for projective geometries (Faure and Frölicher, 1993, 1994), and delivers the mathematical context that embeds the proof of Piron's representation theorem (Piron, 1964, 1976). On the contrary, we propose a dualization/generalization for the notion of property transition, previously motivated to be a join-preserving map (Pool, 1968; Faure *et al.*, 1995; Amira *et al.*, 1998).

2. 'Enriching' $\mathcal{P}arset$-CONCRETE CATEGORIES

For a general overview of the theory of categories we refer to Adamek *et al.* (1990) and Borceux (1994). For quantaloids we refer to Pitts (1988) and Rosenthal (1991).

Definition 1. A *quantaloid* is a category such that:

(i) Every hom-set is a complete lattice; its join is usually denoted by $\vee$.

(ii) Composition of morphisms distributes on both sides over arbitrary joins.

Let $\mathfrak{Q}$ and $\mathfrak{Q}'$ be quantaloids. A *quantaloid morphism* from $\mathfrak{Q}$ to $\mathfrak{Q}'$ is a functor $F: \mathfrak{Q} \to \mathfrak{Q}'$ such that on hom-sets it induces join-preserving maps $\mathfrak{Q}(A, B) \to Q'(FA, FB)$.

For example, $\mathcal{J}\mathcal{C}\mathcal{L}at$ is the quantaloid of complete lattices and join-preserving maps in which the join of maps is computed pointwise. A quantaloid with one object is commonly known as a 'unital quantale'; a quantaloid morphism between two one-object quantaloids is known as a 'unital quantale morphism' (Rosenthal, 1990). Any subcategory of a quantaloid that is closed under the inherited join of morphisms is a subquantaloid. Thus any full subcategory of a quantaloid is a subquantaloid, and selecting from a given quantaloid certain morphisms, but keeping all the objects gives rise to a subquantaloid if and only if the inherited join of morphisms is internal.

Let $\mathcal{P}arset$ denote the category of sets A, B, . . . and partially defined functions $f: A\backslash K \to B$, where $K \subseteq A$ is called the 'kernel' of the partially defined function f, also written as $\ker f$. Then the power-functor is defined as

$$\mathcal{P}: \quad \mathcal{P}arset \to \mathcal{J}\mathcal{C}\mathcal{L}at$$

$$: \begin{cases} A \mapsto 2^A \\ f: A\backslash K \to B \mapsto \mathcal{P}f: 2^A \to 2^B: T \mapsto \{f(t) | t \in T\backslash K\} \end{cases}$$

that is, sets are mapped onto their power-sets, partially defined functions are mapped onto the 'direct image mapping' which is indeed a union-preserving map. This functor is faithful and injective, but neither full nor surjective.

For any $\mathcal{P}arset$-concrete category $\mathcal{A}$, that is, a category that comes equipped with a faithful functor $U\colon \mathcal{A} \to \mathcal{P}arset$, we can compose functors $\mathcal{A} \xrightarrow{U} \mathcal{P}arset \xrightarrow{\mathcal{P}} \mathcal{TCL}at$ and use this to define a category that we shall denote by $\mathcal{PA}$: it has the same objects as $\mathcal{A}$, and for the hom-sets we define

$$\mathcal{PA}(A, B) = \{(\mathcal{P} \circ U)(f) \,|\, f \in \mathcal{A}(A, B)\}$$

So in particular the hom-sets are posets for the pointwise order. Any functor $F\colon \mathcal{A} \to \mathcal{B}$ between two such categories $\mathcal{A} \xrightarrow{U} \mathcal{P}arset$ and $\mathcal{B} \xrightarrow{V} \mathcal{P}arset$ defines a functor

$$\begin{aligned} \mathcal{P}F\colon \quad & \mathcal{PA} \to \mathcal{PB} \\ : \quad & \begin{cases} A \mapsto FA \\ \mathcal{P}Uf\colon 2^{UA} \to 2^{UB} \mapsto \mathcal{P}VFf\colon 2^{VFA} \to 2^{VFB} \end{cases} \end{aligned}$$

which is to say that $\mathcal{P}$ is functorial on the quasicategory of $\mathcal{P}arset$-concrete categories.[3]

Since $\mathcal{PA}(A, B) \subseteq \mathcal{TCL}at(2^{UA}, 2^{UB})$ for any two $\mathcal{PA}$-objects A and B, we can define a category $\mathfrak{Q}^+\mathcal{A}$ with the same objects as $\mathcal{PA}$ (thus the same objects as $\mathcal{A}$), but of which the hom-sets are precisely

$$\mathfrak{Q}^+\mathcal{A}(A, B) = \mathcal{TCL}at(2^{UA}, 2^{UB})$$

Explicitly this means that a morphism $f\colon A \to B$ in $\mathfrak{Q}^+\mathcal{A}$ is determined by an underlying union-preserving map $f\colon 2^{UA} \to 2^{UB}$. So $\mathfrak{Q}^+\mathcal{A}$ is a quantaloid with respect to the pointwise union of maps. Second, by $\mathfrak{Q}^-\mathcal{A}$ we shall denote the category with the same objects as $\mathcal{PA}$ (thus the same objects as $\mathcal{A}$ and as $\mathfrak{Q}^+\mathcal{A}$), but of which the hom-sets are the complete lattices that one obtains if one closes the $\mathcal{PA}$-hom-sets for (arbitrary) pointwise unions of maps: a morphism $f\colon A \to B$ in $\mathfrak{Q}^-\mathcal{A}$ is thus determined by an underlying map $\cup_i \mathcal{P}\mathcal{U}f_i$, the join of maps being their pointwise union, for a set of $\mathcal{A}$-morphisms $\{f_i\colon A \to B\}_i$. If $\mathcal{A}$ is a category in which every hom-set contains a nonzero element—that is, for any two objects A,B of $\mathcal{A}$ there is an $f \in \mathcal{A}(A, B)$ such that $\mathrm{ker}Uf \neq \mathrm{dom}Uf$—then any hom-set of $\mathcal{PA}$ contains a nonzero map, thus any hom-set of $\mathfrak{Q}^-\mathcal{A}$ is a complete lattice (the condition on the hom-sets of $\mathcal{A}$ makes sure that any hom-set of $\mathfrak{Q}^-\mathcal{A}$ contains at least distinct bottom and top which we require for any complete lattice), so $\mathfrak{Q}^-\mathcal{A}$ is a

[3] Remark that $\mathcal{S}et$ is the subcategory of $\mathcal{P}arset$ with the same objects, but all morphisms of which have an empty kernel, and that the domain restriction of $\mathcal{P}$ to $\mathcal{S}et$ yields that, for any morphism $f\colon A \to B$ in $\mathcal{S}et$, $\mathcal{P}f\colon 2^{UA} \to 2^{UB}$ is a union-preserving map that maps only the empty set on the empty set: $(\mathcal{P}f)(T) = \emptyset \Leftrightarrow T = \emptyset$.

quantaloid with respect to pointwise union of morphisms. For a functor F: $\mathscr{A} \to \mathscr{B}$ between two such categories $\mathscr{A} \xrightarrow{U} \mathscr{P}arset$ and $\mathscr{B} \xrightarrow{V} Parset$, we can define a corresponding quantaloid morphism

$$\mathfrak{Q}^-F: \quad \mathfrak{Q}^-\mathscr{A} \to \mathfrak{Q}^-\mathscr{B}$$

$$: \begin{cases} A \mapsto FA \\ \cup_i \mathscr{P}Uf_i: 2^{UA} \to 2^{UB} \mapsto \cup_i \mathscr{P}VFf_i: 2^{VFA} \to 2^{VFB} \end{cases}$$

which translates the idea that the construction of $\mathfrak{Q}^-\mathscr{A}$ is functorial. Note that no such functor can exist for the $\mathfrak{Q}^+$ case because in $\mathfrak{Q}^+\mathscr{A}$ there may be morphisms for which there exists no connection at all with $\mathscr{A}$-morphisms! So we could say that $\mathfrak{Q}^-\mathscr{A}$ is a 'functorial' enrichment of $\mathscr{A}$, and $\mathfrak{Q}^+\mathscr{A}$ merely a 'contextual' enrichment.

By construction, $\mathfrak{Q}^-\mathscr{A}$ is a subquantaloid of $\mathfrak{Q}^+\mathscr{A}$ since the hom-sets of the former are join-sublattices of the hom-sets of the latter (they have the same objects). For an important class of categories this inclusion is strict: we will consider a category $\mathscr{A}$ of which the objects are bounded posets (partially ordered sets with a greatest element 1 and a least element 0, these elements being different) and the morphisms are isotone mappings that map least elements onto least elements (that is, $0 \mapsto 0$), together with a forgetful functor

$$U: \quad \mathscr{A} \to \mathscr{P}arset$$

$$: \begin{cases} (A, \leq) \mapsto A_0 \\ f: (A, \leq) \to (B, \leq) \mapsto Uf: A_0 \backslash K \to B_0: t \mapsto f(t) \end{cases}$$

where $K = \{t \in A_0 | f(t) = 0\} = \mathrm{ker}Uf$ and $A_0 = A \backslash \{0\}$. From now on, we will always assume that every hom-set has at least one nonzero element, to assure that the construction $\mathfrak{Q}^-\mathscr{A}$ yields a quantaloid.

Proposition 1. For such a category $\mathscr{A}$, $\mathfrak{Q}^-\mathscr{A} = \mathfrak{Q}^+\mathscr{A}$ if and only if all objects of $\mathscr{A}$ are 2-chains.

Proof. If $\mathscr{A}$ contains an object $(A, \leq)$ in which there exists $0 < a < 1$, then

$$f: \quad 2^{A_0} \to 2^{A_0}: \qquad \begin{cases} T \mapsto \{a\} \Leftrightarrow a \in T \\ T \mapsto \emptyset \Leftrightarrow a \notin T \end{cases}$$

is trivially a union-preserving map, thus $f \in \mathfrak{Q}^+\mathscr{A}(\mathrm{A}, \mathrm{A})$. For this f to be a morphism of $\mathfrak{Q}^-\mathscr{A}$ as well, there must be a set of $\mathscr{A}$-morphisms h_i: $A \to A$ such that precisely $f = \cup_i \mathscr{P}Uh_i$:

$$\forall T \in 2^{A_0}: \quad f(T) = \cup_i \{Uh_i(t) | t \in T \backslash \mathrm{ker}Uh_i\}$$

Since $f(\{1\}) = \emptyset$, we must have that $h_i(1) = 0$ for all those h_i, but on the other hand $f(\{a\}) = \{a\}$, which means that at least one h_k is such that $h_k(a) = a$. But this contradicts the isotonicity of the $\mathcal{A}$-morphism h_k. If on the contrary the category $\mathcal{A}$ contains only posets which are (isomorphic to) 2-chains, then it is merely an observation that the functorial and the contextual enrichment are the same. ■

As an example, consider

$$U: \quad \mathcal{TCL}at \to \mathcal{P}arset$$

$$: \quad \begin{cases} (L, \vee) \mapsto L_0 \\ f: (L, \vee) \to (M, \vee) \mapsto Uf: L_0 \backslash \mathrm{ker} Uf \to M_0: t \mapsto f(t) \end{cases}$$

With respect to this faithful functor we have the following categories, all three with as objects complete lattices:

- $\mathcal{PTCL}at$: a morphism $f: (L, \vee) \to (M, \vee)$ corresponds to the image $\mathcal{P}Uf: 2^{L_0} \to 2^{M_0}$ of a join-preserving map $f: L \to M$ (thus the underlying map preserves unions).
- $\mathfrak{Q}^-\mathcal{TCL}at$: a morphism $f: (L, \vee) \to (M, \vee)$ corresponds to a map $f: 2^{L_0} \to 2^{M_0}$ that can be written as the pointwise union of $\mathcal{PTCL}at$-maps (then it automatically preserves unions).
- $\mathfrak{Q}^+\mathcal{TCL}at$: a morphism $f: (L, \vee) \to (M, \vee)$ corresponds to a map $f: 2^{L_0} \to 2^{M_0}$ that preserves unions.

As a corollary of Proposition 1, the quantaloid inclusion

$$\mathfrak{Q}^-\mathcal{TCL}at \hookrightarrow \mathfrak{Q}^+\mathcal{TCL}at$$

is strict. In the next section we introduce the notion of 'state transition' as morphism between complete lattices, and show that these can organize themselves in a quantaloid that lies between $\mathfrak{Q}^-\mathcal{TCL}at$ and $\mathfrak{Q}^+\mathcal{TCL}at$.

3. 'STATE TRANSITIONS' AS MORPHISMS BETWEEN COMPLETE LATTICES

Given two complete lattices $(L, \vee)$, $(M, \vee)$ and a map $f: 2^{L_0} \to 2^{M_0}$ that preserves unions, it is easy to verify that if there exists a map $g: L \to M$ that makes

$$\begin{array}{ccc} L & \xrightarrow{g} & M \\ \uparrow\vee & & \uparrow\vee \\ 2^{L_0} & \xrightarrow{f} & 2^{M_0} \end{array}$$

commute (the vertical uparrows denoting, for the respective lattices, the maps

$T \mapsto \vee T$), then this map g is a unique $\mathcal{TCL}at$-morphism. Therefore the following definition makes sense:

Definition 2. Given a subcategory $\mathcal{A}$ of $\mathcal{TCL}at$ and two of its objects $(L, \vee)$, $(M, \vee)$, a union-preserving map $f: 2^{L_0} \to 2^{M_0}$ is called *state transition with respect to* $\mathcal{A}$ if there exists an $\mathcal{A}$-morphism $f_{pr}: L \to M$ that makes the diagram

$$\begin{array}{ccc} L & \xrightarrow{f_{pr}} & M \\ \uparrow\vee & & \uparrow\vee \\ 2^{L_0} & \xrightarrow{f} & 2^{M_0} \end{array}$$

commute. This (unique) morphism f_{pr} is then called the *property transition corresponding to f.*

For a given subcategory $\mathcal{A}$ of $\mathcal{TCL}at$ all morphisms of $\mathcal{PA}$ are state transitions with respect to $\mathcal{A}$:

Proposition 2. For every $\mathcal{A}$-morphism f we have that $(\mathcal{P}Uf)_{pr} = f$. For every set $\{f_i\}_{i \in I}$ of parallel $\mathcal{A}$-morphisms we have that $(\cup_i \mathcal{P}Uf_i)_{pr} = \vee_i f_i$, these joins being computed pointwise.

Proof. For $f:(L, \vee) \to (M, \vee)$ in $\mathcal{A}$ the direct image map $\mathcal{P}Uf: 2^{L_0} \to 2^{M_0}$ is union-preserving, and for any $T \in 2^{L_0}$: $\vee \mathcal{P}Uf(T) = \vee\{Uf(t) | t \in T \backslash \mathrm{ker} Uf\} = \vee\{f(t) | t \in T\} = f(\vee T)$. The second assumption can be proven likewise. ■

This proposition does not say that any $\mathcal{Q}^-\mathcal{A}$-morphism, which is by definition an arbitrary pointwise union of $\mathcal{PA}$-morphisms, is a state transition with respect to $\mathcal{A}$, for it may very well be that, even with all f_i in $\mathcal{A}$, the corresponding property transition $(\cup_i \mathcal{P}Uf_i)_{pr} = \vee_i f_i$ is not an $\mathcal{A}$-morphism. So in general any $\mathcal{Q}^-\mathcal{A}$-morphism is a state transition only with respect to $\mathcal{TCL}at$. But it follows immediately that:

Corollary 1. Every $\mathcal{Q}^-\mathcal{A}$-morphism is a state transition with respect to $\mathcal{A}$ if and only if $\mathcal{A}$ is a subquantaloid of $\mathcal{TCL}at$.

For any state transition f (the subcategory $\mathcal{A}$ is of no importance here) and any $T, T' \in 2^{L_0}$ we have

$$\vee T = \vee T' \Rightarrow \vee f(T) = \vee f(T') \tag{1}$$

because $\vee f(T) = g(\vee T) = g(\vee T') = \vee f(T')$. But the converse is also true: any union-preserving map $f: 2^{L_0} \to 2^{M_0}$ that meets Eq. (1) is a state transition with respect to $\mathcal{TCL}at$, since in this case the map

$$f_{pr}: \quad L \to M: \quad t = \vee T \mapsto \vee f(T)$$

is a well-defined $\mathcal{TCL}at$-morphism that makes the square commute. Thus:

Proposition 3. A map $f: 2^{L_0} \to 2^{M_0}$ is a state transition with respect to $\mathcal{TCL}at$ if and only if it preserves unions and meets the condition of Eq. (1).

In fact, this proposition was taken as definition for 'state transition (with respect to $\mathcal{TCL}at$)' in Coecke and Stubbe (1999a, b) and, under a slightly different form, in Amira *et al.* (1998). Remark that the conditions in this proposition are also necessary for f to be a state transition with respect to $\mathcal{A}$, a subcategory of $\mathcal{TCL}at$, but in general not sufficient. This will prove its use in Proposition 4.

By pasting together commuting diagrams, it is easily seen that the composition of any two state transitions (with respect to a certain $\mathcal{A}$) is again a state transition (with respect to that same $\mathcal{A}$) with corresponding property transition $(g \circ f)_{pr} = g_{pr} \circ f_{pr}$. Trivially also the diagram with identities commutes, thus identities on power-sets are state transitions (with respect to whatever $\mathcal{A}$) with corresponding property transition $(id_{2^{L_0}})_{pr} = id_L$:

$$\begin{array}{ccccc} L & \xrightarrow{f_{pr}} & M & \xrightarrow{g_{pr}} & N \\ \uparrow\vee & & \uparrow\vee & & \uparrow\vee \\ 2^{L_0} & \xrightarrow{f} & 2^{M_0} & \xrightarrow{g} & 2^{N_0} \end{array} \qquad \begin{array}{ccc} L & \xrightarrow{id_L} & L \\ \uparrow\vee & & \uparrow\vee \\ 2^{L_0} & \xrightarrow{id_{2^{L_0}}} & 2^{L_0} \end{array}$$

This means that for a given subcategory $\mathcal{A}$ of $\mathcal{TCL}at$ we can define a category $\mathcal{Q}^{st}\mathcal{A}$ with the same objects as $\mathcal{A}$, in which a morphism $f: (L, \vee) \to (M, \vee)$ is determined by an underlying state transition with respect to $\mathcal{A}$. This category contains $\mathcal{PA}$ (by Proposition 2) and is in turn a subcategory of $\mathcal{Q}^{+}\mathcal{A}$:

$$\mathcal{PA} \hookrightarrow \mathcal{Q}^{st}\mathcal{A} \hookrightarrow \mathcal{Q}^{+}\mathcal{A}$$

The foregoing is best summarized by the functor

$$F_{pr}: \quad \mathcal{Q}^{st}\mathcal{A} \to \mathcal{A}: \quad \begin{cases} (L, \vee) \mapsto (L, \vee) \\ \qquad f \mapsto f_{pr} \end{cases}$$

which is full because $F_{pr} \circ \mathcal{P}: \mathcal{A} \to \mathcal{A}$ is the identity (see Proposition 2); this fact expresses explicitly the duality of state transitions and property transitions (with respect to $\mathcal{A}$).

4. STATE TRANSITIONS AND QUANTALOIDS

If, moreover, $\mathcal{A}$ is a subquantaloid of $\mathcal{TCL}at$, then commutation of the left diagram for each $i \in I$ implies commutation of the right diagram:

$$\begin{array}{ccc} L & \xrightarrow{f^{i}_{,pr}} & M \\ \uparrow\vee & & \uparrow\vee \\ 2^{L_0} & \xrightarrow{f_i} & 2^{M_0} \end{array} \qquad \begin{array}{ccc} L & \xrightarrow{\vee_i f_{i,pr}} & M \\ \uparrow\vee & & \uparrow\vee \\ 2^{L_0} & \xrightarrow{\cup_i f_i} & 2^{M_0} \end{array}$$

since these joins of maps are computed pointwise, which is to say that, if $\mathscr{A}$ is a subquantaloid of $\mathscr{TCL}at$, then $\mathfrak{Q}^{st}\mathscr{A}$ is a quantaloid and the functor F_{pr}: $\mathfrak{Q}^{st}\mathscr{A} \to \mathscr{A}$ is a full quantaloid morphism. $\mathfrak{Q}^{st}\mathscr{A}$ is then a subquantaloid of $\mathfrak{Q}^{+}\mathscr{A}$, and it contains $\mathfrak{Q}^{-}\mathscr{A}$ as subquantaloid (by virtue of Corollary 1 and the fact that in all three quantaloids joins of morphisms are computed as pointwise unions):

$$\mathfrak{Q}^{-}\mathscr{A} \hookrightarrow \mathfrak{Q}_{st}\mathscr{A} \hookrightarrow \mathfrak{Q}^{+}\mathscr{A}$$

Next we point out in detail the relations among these three different constructions.

Proposition 4. For a subcategory $\mathscr{A}$ of $\mathscr{TCL}at$, $\mathfrak{Q}^{st}\mathscr{A} = \mathfrak{Q}^{+}\mathscr{A}$ if and only if all objects of $\mathscr{A}$ are 2-chains.

Proof. If $\mathscr{A}$ contains an object in which there is a chain, $0 < a < 1$. With the "same" counterexample as in the proof of Proposition 1 it can be seen that $\vee f(\{1\}) = \vee\emptyset = 0 \neq a = \vee\{a\} = \vee f(\{a, 1\})$, although $\vee\{1\} = \vee\{a, 1\}$, which contradicts Proposition 3, such that f cannot be a state transition with respect to $\mathscr{TCL}at$ and *a fortiori* f cannot be a state transition with respect to $\mathscr{A}$, a subcategory of $\mathscr{TCL}at$. Conversely, if $\mathscr{A}$ contains only 2-chains, then it is trivial that $\mathfrak{Q}^{st}\mathscr{A} = \mathfrak{Q}^{+}\mathscr{A}$. ∎

Proposition 5. For a subcategory $\mathscr{A}$ of $\mathscr{TCL}at$, $\mathfrak{Q}^{-}\mathscr{A} \not\supseteq \mathfrak{Q}^{st}\mathscr{A}$ if there is an $\mathscr{A}$-object $(L, \vee)$ in which there exist elements a, b, c such that $a < b \vee c$, $a \not\leq b$, $a \not\leq c$.

Proof. Consider the map

$$f\colon \quad 2^{L_0} \to 2^{L_0}\colon \quad \begin{cases} \{t\} \mapsto \{t\} \quad \text{for} \quad t \neq b \bigvee c \\ \{b \bigvee c\} \mapsto \{a, b, c\} \\ T \mapsto \cup_{t\in T} f(\{t\}) \end{cases}$$

for which clearly $f_{pr} = id_L$, thus it is a state transition (with respect to whatever $\mathscr{A}$). Let g: $L \to L$ be a join-preserving map that maps $a \mapsto a$; then $\mathscr{P}g \leq f \Rightarrow \mathscr{P}g(\{b \vee c\}) \subseteq f(\{g \vee c\}) \Rightarrow g(b) \vee g(c) \in \{a, b, c\} \Rightarrow g(a) \leq g(b \vee c) = g(b) \vee g(c) \in \{a, b, c\}$ and $g(a) = a \Rightarrow a = g(b) \vee g(c) \Rightarrow g(b) \leq a$ and $g(c) \leq a$. Should $g(b) = a$, then $\mathscr{P}g(\{b\}) = \{a\}$ and thus $\mathscr{P}g \not\leq f$. If on the contrary $g(b) < a$, then $g(c) = a$ because $g(c) < a$ would imply that $g(b) \vee g(c) < a$, but then $\mathscr{P}g(\{c\}) = \{a\}$ and thus $\mathscr{P}g \not\leq f$. Therefore, for any $g \in \mathscr{TCL}at(L, L)$ such that $\mathscr{P}g \leq f$ we necessarily have that $g(a) = 0$. But then it is impossible to ever write a pointwise union $f = \bigvee_i \mathscr{P}g_i$ with $\{g_i\}_i \subseteq \mathscr{TCL}at(L, L)$ because the join on the right-hand side always fails to map $\{a\} \mapsto \{a\}$. ∎

Corollary 2. If $\mathscr{A}$ is a subquantaloid of $\mathscr{TCL}at$ that contains an object in which there exist elements a, b, c such that $a < b \vee c$, $a \not\leq b$, $a \not\leq c$, then both the inclusions of quantaloids $\mathfrak{Q}^{-}\mathscr{A} \hookrightarrow \mathfrak{Q}^{st}\mathscr{A} \hookrightarrow \mathfrak{Q}^{+}\mathscr{A}$ are strict.

As $\mathscr{TCL}at$ is a trivial subquantaloid of $\mathscr{TCL}at$, we have the full quantaloid morphism

$$F_{pr};\quad \mathfrak{Q}^{st}\mathscr{TCL}at \to \mathscr{TCL}at: \quad \begin{cases} (L, \vee) \mapsto (L, \vee) \\ \quad\ \ f \mapsto f_{pr} \end{cases}$$

expressing explicitly the duality between state transitions and property transitions. Since $\mathfrak{Q}^{st}\mathscr{TCL}at$ contains $\mathfrak{Q}^{-}\mathscr{TCL}at$, which in turn contains $\mathfrak{Q}^{-}\mathscr{A}$ for any subcategory $\mathscr{A}$ of $\mathscr{TCL}at$, we can consider $F_{pr}\mathfrak{Q}^{-}\mathscr{A}$, the image of $\mathfrak{Q}^{-}\mathscr{A}$ through this functor: this is the smallest subquantaloid of $\mathscr{TCL}at$ that contais $\mathscr{A}$; it emerges by closing all hom-sets $\mathscr{A}(A, B) \subseteq \mathscr{TCL}at(A, B)$ for arbitrary (pointwise) joins. Evidently, if $\mathscr{A}$ is a subquantaloid of $\mathscr{TCL}at$, then and only then $\mathscr{A} = F_{pr}\mathfrak{Q}^{-}\mathscr{A}$. It can be verified straightforwardly that the assignment $\mathscr{A} \mapsto F_{pr}\mathfrak{Q}^{-}\mathscr{A}$ is functorial; we will denote the corresponding functor (that thus acts on subcategories of $\mathscr{TCL}at$ and functors between these) as $\mathscr{E}$ and we will refer to $\mathscr{E}\mathscr{A}$ as the *preenrichment* of $\mathscr{A}$. Obviously the quantaloid $\mathfrak{Q}^{-}\mathscr{A}$ is included in $\mathfrak{Q}^{-}\mathscr{E}\mathscr{A}$, so we can write the following inclusion of quantaloids as generalization of the previous material:

$$\mathfrak{Q}^{-}\mathscr{A} \hookrightarrow \mathfrak{Q}^{-}\mathscr{E}\mathscr{A} \hookrightarrow \mathfrak{Q}_{st}\mathscr{E}\mathscr{A} \hookrightarrow \mathfrak{Q}^{+}\mathscr{E}\mathscr{A}$$

Using the various previous propositions, we can give conditions for these inclusions to be strict.

5. CONCLUSION AND EXAMPLES

Theorem 1. For any subcategory $\mathscr{A}$ of $\mathscr{TCL}at$ that contains an object in which there exist elements a, b, c such that $a < b \vee c$, $a \not\leq b$, $a \not\leq c$, the quantaloid inclusions $\mathfrak{Q}^{-}\mathscr{E}\mathscr{A} \hookrightarrow \mathfrak{Q}^{st}\mathscr{E}\mathscr{A} \hookrightarrow \mathfrak{Q}^{+}\mathscr{E}\mathscr{A}$ are strict. Here $\mathscr{E}$ stands for the minimal extension of $\mathscr{A}$ to a quantaloid, $\mathscr{A} \mapsto \mathscr{F}_{pr}\mathfrak{Q}^{-}\mathscr{A}$. If, moreover, $\mathscr{A}$ is a subquantaloid of $\mathscr{TCL}_{at}$, then $\mathscr{E}\mathscr{A} = \mathscr{A}$.

The essence of having an inclusion $\mathfrak{Q}^{-}\mathscr{A} \hookrightarrow \mathfrak{Q}^{st}\mathscr{A}$—or forcing it as $\mathfrak{Q}^{-}\mathscr{E}\mathscr{A} \hookrightarrow \mathfrak{Q}^{st}\mathscr{E}\mathscr{A}$—should be understood in the following way: The join of maps in $\mathfrak{Q}^{st}\mathscr{A}$ physically stands for a lack of knowledge on possible state transitions (Amira *et al.*, 1998; Coecke and Stubbe, 1999a). Therefore, any general collection of state transitions should be closed under joins, which in the case of a categorical formulation leads to a quantaloid structure. The inclusion $\mathfrak{Q}^{-}\mathscr{A} \hookrightarrow \mathfrak{Q}^{st}\mathscr{A}$ then follows by Corollary 1.

Let us now apply all this to some particular categories that have applications in physics. Consider the subcategory $\mathscr{TCL}at_{\mathbf{1}}$ of $\mathscr{TCL}at$ introduced in

Coecke and Moore, (1999) with as objects complete lattices L_i with 'fixed' top $\mathbf{1}$ and an element 1_i such that $\forall a_i \neq \mathbf{1}$: $a_i \leq 1_i$, and as morphisms join-preserving maps with $\mathbf{1} \mapsto \mathbf{1}$ or maps of which the image is $\{0\}$. Since it is a subquantaloid of $\mathcal{TCL}at$, being itself also a quantaloid for pointwise order, all the above considerations that apply to $\mathcal{TCL}at$ apply to $\mathcal{TCL}at_{\mathbf{1}}$.

Let $\mathcal{TCAL}at$ be the category of complete atomistic lattices with as morphisms join-preserving maps that assign atoms to atoms or the least element (Faure and Frölicher, 1993, 1994). If we apply $\mathcal{E}$ to this category, we obtain a full subcategory $\mathcal{ETCAL}at$ of $\mathcal{TCL}at$ as preenrichment. This is the minimal extension of $\mathcal{TCAL}at$ that assures that all by $\mathfrak{Q}^-$ induced morphisms are state transitions. Since the category $\mathcal{P}rop$ introduced in Moore (1995) is a full subcategory of $\mathcal{JCAL}at$ by restricting objects complete orthocomplemented atomistic lattices, all the above applies to it, i.e., within this context one should rather consider $\mathcal{EP}rop$.

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